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A THEOREM OF FROBENIUS, A THEOREM OF AMITSUR-LEVITSKI AND COHOMOLOGY THEORY

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By

Bertram Kostant

Technical Report No. 5

Prepared under Contract AF 49(638)-79

Division File No. 3.22

For

Mathematics Division
Air Force Office of Scientific Research, ARDC,
Washington 25, D. C.

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Department of Mathematics
University of California
Berkeley 4, California

July 1958

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A Theorem of Frobenius, a Theorem of Amitsur-Levitski and Cohomology Theory

by Bertram Kostant

1. Introduction. 1. Let U be a complex n-dimensional vector space and $G = G(U)$ the group of all non-singular linear transformations on U. If ${}^rU = \bigotimes^r U$ is the space of r-tensors over U, then it is a well known theorem that the group G and the symmetric group $\mathcal{S} = \mathcal{S}_r$ both admit natural representations on ${}^rU$ and that the algebras generated by the two image groups are each other's commutators. This theorem plays a large role in Weyl's treatment of the classical groups as given in [5]. One of the most important, yet lesser known, applications is in essence, that it leads to an identification of the cohomology group $H^r(\mathfrak{g})$ with an explicit subspace of the group algebra I of $\mathcal{S}$. Here $\mathfrak{g} = \mathfrak{g}(U)$ is the Lie algebra of all linear transformations on U so that $H^r(\mathfrak{g})$ is the same as the r^{th} cohomology group (with complex coefficients) of the $n \times n$ unitary group. Weyl uses this identification to obtain a proof of the fact that

$$(1 + t)(1 + t^3)\cdots(1 + t^{2n-1}) \qquad (1.1.1)$$

is the Poincaré polynomial of the $n \times n$ unitary group.

2. Our investigation originated with the desire to go further than this in the following respect. We wanted to find explicity those operators on ${}^rU$ made to correspond

to the primitive elements of $H^r(\mathfrak{g})$ by the identification above (the subspace of I corresponding to $H^r(\mathfrak{g})$ is faithfully represented on ${}^rU$). As it turned out, results completely unexpected were obtained. This paper contains only a partial account of these results.

If r is even, there is no non-zero primitive class in $H^r(\mathfrak{g})$. If r is odd, say $r = 2p - 1$, where $1 \leqq p \leqq n$, then the space $P^r(\mathfrak{g})$ of primitive elements is one-dimensional. Hence in such a case there is a single operator D on ${}^rU$ to be sought. Consider the usual direct sum decomposition ${}^rU = \Sigma_\alpha U_\alpha$ of ${}^rU$, indexed by the partitions, α, of r having n or less parts. An early result (obtained through the theory of self-associated partitions) showed that D vanishes on all the spaces U_α, save one, U_ϵ, and that D^2 is a non-zero scalar on U_ϵ. Here ϵ is the simplest of all self-associated partitions of r. Its graph is given in §4.8. (In this paper this result is Theorem 5.4 asserting that $D = B(h^o_\epsilon)$ where $h^o_\epsilon \in I$ is given by (5.4.1). On the other hand, there was a natural candidate, $B(h_\epsilon)$, (see §4.6) for the operator D. The operator $B(h_\epsilon)$ is the one used by Weyl to obtain the Poincaré polynomial. The fact that $B(h_\epsilon)$ is indeed D (as we later proved) then yielded each of the following three, seemingly unrelated, known (but unknown to us at the time) theorems. Conversely, each of these theorems, as we shall see, implies $B(h_\epsilon) = D$.

I. The theorem of Amitsur-Levitski on the minimal standard identity satisfied by the general matrix algebra over any field. (See [1].)

II. The formula of Dynkin on the invariant cocycles of $\mathcal{g}$ representing its primitive cohomology classes. (See [7], p. 141.)

III. A theorem of Frobenius applied to the characters of the alternating group $\mathcal{a} = \mathcal{a}_r$ corresponding to the partition ϵ. Properly interpreted this takes the form $h_\epsilon^o = h_\epsilon$. (III is a special case of a general theorem of Frobenius on the characters of $\mathcal{a}$ corresponding to any self-associated partition. See [4] or [3], p. 185.)

From the perspective gained here the statements $B(h_\epsilon) = D$, I, II and III appear as different ways of expressing the same idea. This, we think, will become clear from the proofs of equivalence of these statements. See especially §5 in this regard. The biggest gap is between the statements I and II on the one hand and the other two statements on the other. The gap, however, is bridged by Lemma 4.9 which, we later discovered, is a generalization of a result of Schur. See [8], Hilfssatz III.

In view of the emphasis put on the equivalence of I, II and III we feel somewhat called upon to defend this emphasis. In our opinion it seems safe to say that neither I, II nor III, at least as they appear in the literature today, can be regarded as easy theorems. The theorem of Amitsur-Levitski was conjectured for some time by a number

of people before it was finally proved. See e.g. Math. Rev. regarding [1]. As to the original proof see the remark in [6], p. 228. The original proof of the formula of Dynkin depends upon the highly sophisticated transgression theory of Chevalley and Cartan. A proof of the Frobenius theorem has recently appeared in [3]. Even after considerable preparation for it, the proof requires five pages of very subtle arguments. Incidentally, we make no claim that the proofs of these results given here are simplications. It is only the novelty of their relation to each other and to the finding of D that is being brought out.

3. In §2 we recall some of the basic theorems in the homology and cohomology theory of Lie algebras which will be needed in §§3 and 5. Also included (Lemma 2.6) is a simple but useful criterion for the primitivity of a cohomology class.

In §3 we obtain the Amitsur-Levitski theorem from the formula of Dynkin and the theory of §2. A central role in the proof is played by the theorem of Koszul-Samelson on the behavior of primitive cohomology classes of Lie algebras under homomorphisms. The technique involved in the use of this theorem leads, in a natural way, to a consideration of standard identities for Lie subalgebras $\mathfrak{k}$ of $\mathfrak{g}$. A curious feature of the Poincaré polynomial of the rotation group in even dimensions is then exploited to yield a new standard identity (Theorem 3.5). This identity implies that the set of all $n \times n$ skew-symmetric matrices

has a lower degree of commutativity (at least 2 less) than does $\mathfrak{gl}(n, C)$ itself when n is even.

The condition for an even standard identity to hold for a general Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is then put in cohomological terms (Theorem 3.6). One expresses the condition in terms of the filtration (usually used to define the spectral sequence associated with $(\mathfrak{k}, \mathfrak{g}, \mathfrak{g}/\mathfrak{k})$ of the differential algebra $C^*(\mathfrak{g})$ defined by the subalgebra $\mathfrak{k}$.

§4 is devoted to the role played by the Frobenius theorem. The equivalence of I and III is proved there. The development rests upon the theory of Young diagrams and no use of, or application to, cohomology is made.

In §5, $H^r(\mathfrak{g})$ is "realized" as operators on ${}^rU$. The formula of Dynkin is then an immediate consequence of the Frobenius theorem, Lemma 4.9 and Lemma 2.6. A second application of the Frobenius theorem leads to a sharpening of a result of Weyl (see [5], pp. 235-238). Next, by considering "cup products (Lemma 5.6) of operators," a proof of II is given independent of I and III. Finally, the relation (Theorem 5.6) between the shape of the various self-associated Young diagrams and the cup products of the primitive classes of $\mathfrak{g}$ is given.

2. Preliminaries on cohomology theory. For the purposes of greater clarity (and to establish notation) we give a brief review of homology and cohomology theory of Lie algebras as developed in [2].

Let $\mathfrak{a}$ be a Lie algebra (dim n) over any field K having characteristic different from 2. Let E designate the underlying vector space of $\mathfrak{a}$. Let E^* be the dual to E.

We recall that the duality between a vector space E and its dual E^* is extended to a duality between their respective exterior algebras $\wedge E$ and $\wedge E$ as follows: The subspace $\wedge^p E$ of p-vectors over E is orthogonal to the space $\wedge^q E^*$ of q-vectors over E^* if $p \neq q$, and if $p = q$ then the duality may be given by

$$\langle x^1 \wedge \cdots x^i \wedge \cdots x^p,\ g^1 \wedge \cdots g^i \wedge \cdots g^p \rangle = \det |\langle x^i, g^i \rangle| \quad (2.1.1)$$

where $x^i \in E$, $g^i \in E^*$, $i, j = 1, 2, \cdots, p$.

2. Now one introduces a boundary operator ∂ in $\wedge E$ where $\partial : \wedge^p E \to \wedge^{p-1} E$, $p = 1, 2, \cdots, n$. The operator ∂ vanishes on $\wedge^0 E = K$ and may be defined on $\wedge^p E$ as follows:

$$\partial(x^1 \wedge \cdots x^i \wedge \cdots x^j \wedge \cdots x^p) = \sum_{i<j} (-1)^{i+j+1} [x^i, x^j] \wedge x^1 \wedge \cdots \hat{x}^i \wedge \cdots \hat{x}^j \wedge \cdots x^p.$$

One has that $\partial^2 = 0$. The algebra $\wedge E$ furnished with the operator ∂ will be designated by $C_*(\mathfrak{a})$ and its elements are called chains. In particular, the elements of the space $\wedge^p E$, written now as $C_p(\mathfrak{a})$, are the p-chains. The zeros of ∂ are of course the cycles and one constructs, in the usual way, the homology space $H_*(\mathfrak{a})$.

The differential operator δ in $\wedge E^*$ is defined as the transpose of $-\partial$. The operator δ is an anti-derivation and $\wedge E^*$ together with this operator is a graded differential algebra in the sense of H. Cartan. One designates it by $C^*(\sigma)$ and the elements of the space $\wedge^p E^*$ (now written as $C^p(\sigma)$) are called the p-cochains. The zeros of δ are called the cocycles. In this case, however, one obtains $H^*(\sigma)$ as the cohomology algebra. The multiplicative structure (cup product) in $H^*(\sigma)$ is a consequence of the fact that δ is an anti-derivation. If a, b are cocycles in $C^*(\sigma)$ and $\underline{a}$, $\underline{b}$ are their respective classes in $H^*(\sigma)$ then the cup product $\underline{a} \cup \underline{b}$ is the class of $a \wedge b$.

[Note: As in [2] we adopt the convention of designating the homology (resp. cohomology) class of a cycle (resp. cocycle) with the same letter used to designate the cycle (resp. cocycle). They are distinguished in that the letter for the homology (resp. cohomology) class appears in boldface.]

Now the duality between $C_*(\sigma)$ and $C^*(\sigma)$ induces a duality between $H_*(\sigma)$ and $H^*(\sigma)$. Furthermore, analogous to the duality relation between $C_p(\sigma)$ and $C^q(\sigma)$ mentioned before, one has that the orthogonal complement of $H_p(\sigma)$ is $\sum_{q \neq p} H^q(\sigma)$.

3. For any $X \in \sigma$ let $\theta_*(X)$ be the operator on $C_*(\sigma)$ defined by extending ad $X: \sigma \to \sigma$ as a derivation of $C_*(\sigma)$. Of course $X \to \theta_*(X)$ is then a representation of σ on $C_*(\sigma)$.

Similarly $X \to \theta^*(X)$ defines a representation of $\mathfrak{a}$ on $C^*(\mathfrak{a})$ as derivations if one defines $\theta^*(X)$ as the negative transpose of $\theta_*(X)$. A chain (resp. cochain) is called invariant if it is annihilated by all $\theta_*(X)$ (resp. $\theta^*(X)$). The algebra of invariant chains (resp. cochains) will be designated by $J_*(\mathfrak{a})$ (resp. $J^*(\mathfrak{a})$).

We now assume that $\mathfrak{a}$ is reductive (i.e., the representation $\theta_*(X)$ is completely reducible). In such a case we may introduce multiplication (Pontrjagin product) in $H_*(\mathfrak{a})$. Here $\underline{u} \cup \underline{v}$ is defined to be the class of $u \wedge v$ when u and v are representative cycles of $\underline{u}$ and $\underline{v}$ respectively, chosen so that $u \wedge v$ is again a cycle. Furthermore, any invariant chain (resp. cochain) is a cycle (resp. cocycle) and if $J_*(\mathfrak{a}) \to H_*(\mathfrak{a})$ (resp. $J^*(\mathfrak{a}) \to H^*(\mathfrak{a})$) is the mapping $u \to \underline{u}$ (resp. $a \to \underline{a}$), then this mapping defines an isomorphism of the respective algebras. If $u \in J_*(\mathfrak{a})$, $a \in J^*(\mathfrak{a})$ it follows easily that

$$\langle u, a \rangle = \langle \underline{u}, \underline{a} \rangle \qquad (2.3.1)$$

4. Now let $D_*(\mathfrak{a})$, the space of decomposable classes, be the subspace of $H_*(\mathfrak{a})$ defined as the linear span of all elements of the form $\underline{a} \cup \underline{b}$ where $\underline{a}$ and $\underline{b}$ are elements of positive degree. One defines $D^*(\mathfrak{a}) \subset H^*(\mathfrak{a})$ analogously. The space $P_*(\mathfrak{a}) \subset H_*(\mathfrak{a})$ (resp. $P^*(\mathfrak{a}) \subset H^*(\mathfrak{a})$) of primitive elements is then defined as the orthocomplement to $D^*(\mathfrak{a})$ (resp. $D_*(\mathfrak{a})$).

Now assume the field K has characteristic zero. Then the following theorem of Koszul generalizes a theorem of

Samelson.

Theorem 2.4 (Koszul). The injection map of $P_*(\mathfrak{a})$ (resp. $P^*(\mathfrak{a})$) into $H_*(\mathfrak{a})$ (resp. $H^*(\mathfrak{a})$) extends to an isomorphism of the algebra $\Lambda(P_*(\mathfrak{a}))$ (resp. $\Lambda(P^*(\mathfrak{a}))$ onto the algebra $H_*(\mathfrak{a})$ (resp. $H^*(\mathfrak{a})$).

Now let i be the linear isomorphism of $\Lambda(P^*(\mathfrak{a}))$ onto the dual of $\Lambda(P_*(\mathfrak{a}))$ given by the duality of $H_*(\mathfrak{a})$ and $H^*(\mathfrak{a})$ and the isomorphisms above. Then if the dual to $\Lambda P_*(\mathfrak{a}))$ is identified with $\Lambda(P(\mathfrak{a}))^*$ in the sense of §2.1 the linear isomorphism

$$i: \Lambda(P^*(\mathfrak{a})) \to (P(\mathfrak{a}))^*$$

is an algebraic isomorphism of exterior algebras mapping $P^*(\mathfrak{a})$ onto $(P(\mathfrak{a}))^*$.

In other words, identifying $H_*(\mathfrak{a})$ and $H^*(\mathfrak{a})$, respectively, with $\Lambda P_*(\mathfrak{a})$ and $\Lambda P^*(\mathfrak{a})$ by virtue of the first part of above theorem the second part asserts that the consequent duality between $\Lambda(P_*(\mathfrak{a}))$ and $\Lambda(P^*(\mathfrak{a}))$ is just the usual duality between exterior algebras given in §2.1, that is, after it is recognized that $P^*(\mathfrak{a})$ is in a natural way isomorphic to $(P(\mathfrak{a}))^*$.

5. Now let $\mathfrak{a}$ and $\mathfrak{b}$ be two reductive Lie algebras over a field F of characteristic zero and let $\varphi: \mathfrak{b} \to \mathfrak{a}$ be a homomorphism. Denote by $\varphi_*, \varphi_*: H_*(\mathfrak{b}) \to H_*(\mathfrak{a})$, and $\varphi^*, \varphi^*: H^*(\mathfrak{a}) \to H^*(\mathfrak{b})$, the respective homomorphisms of the homology and cohomology induced by φ.[1] The proof of Theorem 10.3, [2], asserts that both φ^* and φ_* carry primitive ele-

[1] As usual one replaces the * by k when dealing with homogeneous elements of degree k.

ments into primitive elements. That is,

Theorem 2.5 (Koszul-Samelson). Let $\mathfrak{a}$ and $\mathfrak{k}$ be two reductive Lie algebras over a field F of characteristic 0 and let $\varphi:\mathfrak{k}\to\mathfrak{a}$ be a homomorphism. Then

$$\varphi_*: P_*(\mathfrak{k}) \to P_*(\mathfrak{a})$$

and

$$\varphi^*: P^*(\mathfrak{a}) \to P^*(\mathfrak{k}).$$

6. Primitivity will play an important role for us. The following lemma provides a simple and useful (from our point of view--see §5.4) criterion that a cohomology class be primitive.

Lemma 2.6. Let $\mathfrak{a}$ and $\mathfrak{k}$ be two reductive algebras over a field of characteristic zero. Let

$$\varphi:\mathfrak{k}\to\mathfrak{a}$$

be a homomorphism. If

$$\varphi^i: H^i(\mathfrak{a}) \to H^i(\mathfrak{k})$$

is an isomorphism for $i = 0, 1, \cdots, k-1$ and $\underline{b} \in H^k(\mathfrak{a})$ is in the kernel of φ^k then $\underline{b} \in P^k(\mathfrak{a})$.

Proof: If φ^i is an isomorphism for $i = 0, 1, \cdots, k-1$ then by duality it follows that φ_i is onto for $i = 0,1,\cdots,k-1$. But then for any two classes $\underline{u}$, $\underline{v} \in H_*(\mathfrak{a})$ of degree less than k there exist classes $\underline{u}'$, $\underline{v}' \in H_*(\mathfrak{k})$ such that $\varphi_*(\underline{u}') = \underline{u}$ and $\varphi_*(\underline{v}') = v$. But then

$$\varphi_*(\underline{u}' \cup \underline{v}') = \underline{u} \cup \underline{v}.$$

But since $\varphi^*(\underline{b}) = 0$ this implies

$$\langle \underline{u} \cup \underline{v}, \underline{b} \rangle = 0$$

and hence $\underline{b}$ is orthogonal to $D_*(\mathfrak{a})$. That is, $\underline{b} \in P^*(\mathfrak{a})$. Q.E.D.

Note: It is clear from the proof that Lemma 2.6 still holds if we interchange the roles of homology and cohomology. This yields a corresponding criterion for the primitivity of homology classes of $\mathfrak{h}$.

7. From the point of view of representation theory it is more convenient to deal with complex cohomology over complex Lie algebras rather than real cohomology over real Lie algebras (usual situation). The universal coefficient theorem (see below) when applied to Lie algebras permits us to go back and forth between these two cases.

Assume K has characteristic zero. Let L be a field extension of K and let $\mathfrak{g} = \mathfrak{a} \otimes_K L$. Then $\mathfrak{g}$ is, of course, the Lie algebra obtained from $\mathfrak{a}$ by extending the coefficients from K to L. Now assume $\mathfrak{a}$ is reductive. Then it is easy to verify that $\mathfrak{g}$ is likewise reductive. Indeed (see §9, [2]), $\mathfrak{a}$ is a direct product of a semi-simple Lie algebra and an abelian Lie algebra. This property is, of course, preserved under field extensions and every such algebra is reductive.

Now the universal coefficient theorem asserts that

$$H^*(\mathfrak{g}) = H^*(\mathfrak{a}) \otimes_K L$$

and (2.7.1)

$$H_*(\mathfrak{g}) = H_*(\mathfrak{a}) \otimes_K L.$$

Indeed, it is trivial to verify (2.7.1) and the following relations directly:

$$P^*(\mathfrak{g}) = P^*(\mathfrak{a}) \otimes_K L$$

and (2.7.2)

$$P_*(\mathfrak{g}) = P_*(\mathfrak{a}) \otimes_K L.$$

3. Cohomology and the standard identity. 1. For more details on the subject matter of the following section see [6], Chapter X. In particular see §§4-6, pp. 224-230, on multilinear identities and their relation to the theory of P.I. algebras (algebras with a polynomial identity).

2. We use here the terminology of [6]. Let A be an associative algebra over a field K. Let $x_1, x_2, \cdots, x_n$ be elements of A. Define

$$[x_1 x_2 \cdots x_n] = \sum \pm x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (3.2.1)$$

where the sum is over all permutations of $1,2,\cdots,n$ and the sign is + or - according as the permutation is even or odd.

The polynomial $[x_1 x_2 \cdots x_n]$ is called the standard polynomial of degree n and in case

$$[x_1 x_2 \cdots x_n] = 0$$

for all $x_1, x_2, \cdots, x_n \in A$ one says that A satisfies the standard identity of degree n.

Now it is obvious that if A satisfies the standard identity of degree n then it satisfies one for every degree greater than n. Hence if A satisfies any standard identity at all, the one of lowest degree takes on significance. This is especially true from the following point of view. Clearly an algebra A is commutative if and only if it satisfies the standard identity of degree 2. In a sense then, the failure of an algebra to satisfy a standard identity of a given degree provides a measure of its non-commutativity.

One is thus led to define the degree of commutativity of an algebra $\mathcal{A}$ to be the minimum degree of the standard identities which it satisfies, that is, in case it satisfies at least one standard identity. Since it is easily seen that every finite dimensional algebra satisfies some standard identity one can always speak of the degree of commutativity of such an algebra.

3. Now consider the case when $\mathcal{A} = \mathcal{g}_n$, the $n \times n$ matrix algebra. It is a simple matter to show that the degree of commutativity of $\mathcal{g}_n$ is not less than $2n$. In fact, it is just as simple to show that $\mathcal{g}_n$ can satisfy no multilinear identity of degree less than $2n$ (see for example Prop. I, p. 226, [6]). On the other hand, curiously enough it has proved exceedingly difficult to determine whether or not $\mathcal{g}_n$ satisfies the standard identity of degree $2n$ or not. In fact, we have been informed that it had been conjectured for some time by many, including I. Kaplansky, F. W. Levi, and J. Levitski, before it was finally proved, by Amitsur and Levitski in [1], that the degree of commutativity of $\mathcal{g}_n$ is $2n$. The question immediately reduces to the case where the x_i of §3.2 are matrix units. The proof of Amitsur-Levitski depends upon an involved inductive argument on the number of x_i which are idempotent.

In the next section we will deduce the Amitsur-Levitski theorem and a new standard identity as a consequence of known facts about the cohomology of the unitary group.

To obtain these identities the Koszul-Samelson theorem is used in a very essential way.

4. Let $\mathfrak{a}$ be the real Lie algebra of all $n \times n$ skew-Hermitian matrices. Being the Lie algebra of the $n \times n$ unitary group the following facts are well known about $P_*(\mathfrak{a})$ and $P^*(\mathfrak{a})$: (1) $\dim P_*(\mathfrak{a}) = \dim P^*(\mathfrak{a}) = n$, (2) one may find a basis $\underline{b}_j$, $j = 1, 3, \cdots, 2n-1$ of $P^*(\mathfrak{a})$, the elements of which are unique up to scalar multiples, such that $\underline{b}_j \in P^j(\mathfrak{a})$. Of course then, if $\underline{u}_j \in P_*(\mathfrak{a})$, $j = 1, 3, \cdots, 2n-1$, is the dual basis one has $\underline{u}_j \in P_j(\mathfrak{a})$.

In [7] Dynkin has shown that (after choosing the scalar multiple properly) the unique invariant cocycle $b_j \in C^j(\mathfrak{a})$ in the class $\underline{b}_j$ may be given by

$$\langle b_j, s_1 \wedge s_2 \wedge \cdots \wedge s_j \rangle = \operatorname{tr} [s_1 s_2 \cdots s_j] \qquad (3.4.1)$$

where the $s_i \in \mathfrak{a}$ and the expression $[s_1 s_2 \cdots s_j]$ is defined in §3.2. The proof of Dynkin is based upon transgression theorems of Cartan and Chevalley (see [9], p. 60) to the effect that the set $P^*(\mathfrak{a})$ as a subset of $H^*(\mathfrak{a})$ is characterized as the range of the map ρ, defined in [9], of the space of symmetric invariants over E^* (see §2.1) into $H^*(\mathfrak{a})$.

Now let $\mathfrak{g} = \mathfrak{g}_n$ be the set of all $n \times n$ complex matrices regarded as a Lie algebra. Then $\mathfrak{g} = \mathfrak{a} \otimes C$ is, of course, the complexification of $\mathfrak{a}$. By (2.7.1) and (2.7.2)

$$H^*(\mathfrak{g}) = H^*(\mathfrak{a}) \otimes C$$

and

$$P^*(\mathfrak{g}) = P(\mathfrak{a}) \otimes C.$$

Hence the elements b_j, $j = 1, 3, \cdots, 2n - 1$, may also be regarded as forming a basis of $P^*(\mathcal{g})$ and for every $A_1, A_2, \cdots, A_j \in \mathcal{g}$ we have

$$\langle b_j, A_1 \wedge A_2 \wedge \cdots \wedge A_j \rangle = \operatorname{tr}[A_1 A_2 \cdots A_j]. \qquad (3.4.2)$$

We need the following simple lemma.

Lemma 3.4. *Let* $A_1, A_2, \cdots, A_m \in \mathcal{g}$. *In case* m *is even*

$$\operatorname{tr}[A_1 A_2 \cdots A_m] = 0$$

and if m *is odd*

$$\operatorname{tr}[A_1 A_2 \cdots A_m] = m \operatorname{tr} A_m [A_1 A_2 \cdots A_{m-1}].$$

Proof. Let $\mathcal{S}_m$ be the permutation group of 1, 2, $\cdots$, m. Let $\mathcal{R}$ be the subgroup generated by the single cycle $\tau_1 = (1, 2, 3, \cdots, m)$ and let $\mathcal{U} \subset \mathcal{S}$ be the subgroup leaving m fixed. Then $\mathcal{R} \cap \mathcal{U} = e$ and $\mathcal{S} = \mathcal{U}\mathcal{R}$. Hence

$$\operatorname{tr}[A_1 A_2 \cdots A_m] = \sum_{\sigma \in \mathcal{U}} \operatorname{sg}(\sigma) \sum_{\tau \in \mathcal{R}} \operatorname{sg}(\tau) \operatorname{tr} A_{\sigma\tau(1)} A_{\sigma\tau(2)} \cdots A_{\sigma\tau(m)}.$$

Since the trace is invariant under the action of $\mathcal{R}$ and since τ_1 is odd if and only if m is even it follows that the second summation vanishes in case m is even and when m is odd;

$$\sum_{\tau \in \mathcal{R}} \operatorname{sg}(\tau) \operatorname{tr} A_{\sigma\tau(1)} A_{\sigma\tau(2)} \cdots A_{\sigma\tau(m)} =$$

$$m \operatorname{tr} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(m)}. \qquad \text{Q.E.D.}$$

Theorem 3.4 (Amitsur-Levitski). *Let* $\mathcal{g}$ *be the* $n \times n$ *matrix algebra over any field* K. *Then if* $A_1, A_2, \cdots, A_{2n}$ *are arbitrary elements of* $\mathcal{g}$

$$[A_1 A_2 \cdots A_{2n}] = 0.$$

Proof: To prove the theorem it suffices to prove it for the case when K is the complex numbers. This follows easily from the fact that a standard identity will hold if and only if it holds for the matrix units e_{ij}. Thus the question is independent of the field.

Now let $\mathfrak{h}$ be the algebra of all $(n+1) \times (n+1)$ complex matrices. Identify $\mathfrak{g}$ with the subalgebra of all matrices with zeros in the last row and last column and let $\varphi: \mathfrak{g} \to \mathfrak{h}$ be the injection homomorphism. Now consider $\varphi^*: H^*(\mathfrak{h}) \to H^*(\mathfrak{g})$. Let $\underline{b}_{2n+1} \in P^{2n+1}(\mathfrak{h})$ be the primitive class given by (3.4.1). Here $\mathfrak{h}$ replaces the $\mathfrak{g}$ of §3.4. Let $\underline{a}_{2n+1} = \varphi^*(\underline{b}_{2n+1})$. We assert $\underline{a}_{2n+1} = 0$. Indeed $\underline{a}_{2n+1} \in H^{2n+1}(\mathfrak{g})$. But by Theorem 2.5, $\underline{a}_{2n+1} \in P^{2n+1}(\mathfrak{g})$. On the other hand, $P^{2n+1}(\mathfrak{g}) = 0$. (The highest primitive class of $\mathfrak{g}$ has degree $2n - 1$. See (1.1.1).) Thus $\underline{a}_{2n+1} = 0$. But this implies

$$\operatorname{tr}[A_1 A_2 \cdots A_{2n+1}] = 0 \qquad (3.4.3)$$

for all $A_1, A_2, \cdots, A_{2n+1} \in \mathfrak{g}$. Let $B = A_{2n+1}$. Then by Lemma 3.4

$$\operatorname{tr} B[A_1 A_2 \cdots A_{2n}] = 0$$

for an arbitrary $B \in \mathfrak{g}$. This, however, implies

$$[A_1 A_2 \cdots A_{2n}] = 0. \qquad \text{Q.E.D.}$$

5. Now it is suggestive from this proof of Theorem 3.4 (we refer mainly to the behavior of $\underline{b}_j$ under homomorphisms) to take up the notion of standard identities and degree of

commutativity for Lie algebras as well as associative algebras, that is, in case the Lie algebra is given as a Lie subalgebra of a matrix algebra. (Presumably one could do this for an abstract Lie algebra; however, dependence or independence upon the various representations of the Lie algebra would have to be settled.)

Assume $\mathfrak{k} \subset \mathfrak{g}$ is a complex Lie algebra of $n \times n$ matrices. Consider the question as to whether or not $\mathfrak{k}$ satisfies the standard identity of degree $2p$, $1 \leq p < n$. That is, whether or not

$$[A_1 A_2 \cdots A_{2p}] = 0 \qquad (3.5.1)$$

for all $A_1, A_2, \cdots, A_{2p} \in \mathfrak{k}$. Clearly if (3.5.1) holds then by Lemma 3.4

$$\operatorname{tr} [A_1 \cdots A_{2p+1}] = 0$$

for all $A_1, A_2, \cdots, A_{2p+1} \in \mathfrak{k}$ and hence if $\varphi: \mathfrak{k} \to \mathfrak{g}$ is the injection homomorphism then $\varphi^*(\underline{b}_{2p+1}) = 0$. Thus the vanishing of $\varphi^*(\underline{b}_{2p+1})$ is a necessary condition for (3.5.1).

Now assume the bilinear form $(A,B) = \operatorname{tr} AB$ is non-singular on $\mathfrak{k}$.[2] We test the sufficiency of the condition above. Assume $\varphi^*(\underline{b}_{2p+1}) = 0$. Then from Lemma 3.4

$$[A_1 A_2 \cdots A_{2p}]$$

is contained in the orthocomplement (relative to (A, B)) of $\mathfrak{g}$ in $\mathfrak{g}$ for every $A_1, A_2, \cdots, A_{2p} \in \mathfrak{k}$. Hence if we knew

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[2] This is, of course, true in the all important case where $\mathfrak{k}$ is the complexification of a "compact" Lie subalgebra.

a priori that $[A_1A_2 \cdots A_{2p}] \in \mathfrak{k}$ for all $A_1, A_2, \cdots, A_{2p} \in \mathfrak{k}$ and $\varphi^*(\underline{b}_{2p+1}) = 0$ we could conclude that $\mathfrak{k}$ satisfies the standard identity of degree 2p. This argument applies in the following example.

Let $n = 2m$. Let $\mathfrak{k} = D_m$ be the Lie algebra of all complex skew-symmetric matrices. Then $\mathfrak{k}$ is the complexification of the Lie algebra of the rotation group and hence by [5], p. 238, there exists the basis $\underline{c}_{2m-1}$, $\underline{d}_j$, $j = 3, 7, \cdots, 4m - 5$, of $P^*(\mathfrak{k})$ where the degree is given by the subscript. Thus if $\varphi: \mathfrak{k} \to \mathfrak{g}$ is the injection homomorphism it follows that φ^* annihilates the spaces $P^j(\mathfrak{g})$ where j takes the values $1, 5, 9, \cdots, 4m - 3$. Also since the highest degree among the $\underline{d}_i$ is (and this is the essential point for us) $4m - 5$ and not $4m - 1$ it follows that φ^* annihilates the class $\underline{b}_{4m-1}$. [Although no use will be made of the following fact it should be recalled that φ^* does not annihilate $\underline{b}_j$ for $j = 3, 7, \cdots, 4m - 5$. In fact, the $\underline{d}_j$ can be chosen so that $\varphi^*(\underline{b}_j) = \underline{d}_j$ for these values of j.]

Now no standard identity will arise from the fact that φ^* annihilates the classes $\underline{b}_j$ for $j = 1, 5, 9, \cdots, 4m-3$. On the other hand as we shall see, a new standard identity will arise from the fact that φ^* annihilates the $4m - 1$ class $\underline{b}_{4m-1}$. To see this we first observe the following

Lemma 3.5. *Let $A_1, A_2, \cdots, A_{2p}$ be any 2p complex skew-symmetric matrices. Then*

$$[A_1 A_2 \cdots A_{2p}]$$

is symmetric if p is even and skew-symmetric if p is odd.

Proof: If $A \in \mathcal{g}$, let A^t designate the transposed matrix. Clearly $(A_1 A_2 \cdots A_m)^t = (-1)^m A_m \cdots A_2 A_1$. Let σ be the permutation $(1, 2p)(2, 2p-1) \cdots (p, p+1)$. Then clearly $sg(\sigma) = (-1)^p$. On the other hand, if $\mathcal{S}$ is the group pf permutations on 1, 2, $\cdots$, 2p then

$$\begin{aligned}
[A_1 A_2 \cdots A_{2p}]^t &= \sum_{\tau \in \mathcal{S}} sg(\tau)(A_{\tau(1)} A_{\tau(2)} \cdots A_{\tau(2p)})^t \\
&= \sum_{\tau \in \mathcal{S}} sg(\tau)\, A_{\tau\sigma(1)} A_{\tau\sigma(2)} \cdots A_{\tau\sigma(2p)} \\
&= (-1)^p \sum_{\tau \in \mathcal{S}} sg(\tau\sigma) A_{\tau\sigma(1)} A_{\tau\sigma(2)} \cdots A_{\tau\sigma(2p)} \\
&= (-1)^p [A_1 A_2 \cdots A_{2p}]. \qquad \text{Q.E.D.}
\end{aligned}$$

Now, as we have seen, $\varphi^*(\underline{b}_{2p+1}) = 0$ when p is even or when $p = 2m - 1$. But Lemma 3.4 asserts then that

$$\operatorname{tr} A_{2p+1}[A_1 A_2 \cdots A_{2p}] = 0 \qquad (3.5.1)$$

for all $A_1, A_2, \cdots, A_{2p+1} \in \mathcal{k}$ for these values of p.

On the other hand, (3.5.1) obviously holds for even p. This follows immediately from Lemma 3.5 since $[A_1 A_2 \cdots A_{2p}]$ is symmetric in this case and tr AB = 0 whenever A is skew-symmetric and B is symmetric. Hence (3.5.1) yields no information for even p. On the other hand, if $p = 2m - 1$ Lemma 3.5 asserts that $[A_1 A_2 \cdots A_{2p}] \in \mathcal{k}$. Since A_{2p+1} is arbitrary in $\mathcal{k}$ the non-singularity of the form (A,B) implies

$$[A_1 A_2 \cdots A_{4m-2}] = 0$$

for all $A_1, A_2, \cdots, A_{4m-2} \in \mathfrak{k}$. Hence $\mathfrak{k}$ satisfies a a standard identity (degree $4m - 2$) which $\mathfrak{g}$ does not satisfy. In summary we have proved

Theorem 3.5. *Let $\mathfrak{k}$ be the Lie algebra of all $n \times n$ complex skew-symmetric matrices. Then if n is even $\mathfrak{k}$ satisfies the standard identity of degree $2n - 2$. That is,*

$$[A_1 A_2 \cdots A_{2n-2}] = 0$$

for all $A_1, A_2, A_{2n-2} \in \mathfrak{k}$.

Note: We may regard Theorem 3.5 as a generalization of the familiar case which occurs when $n = 2$. For $n = 2$ Theorem 3.5 reduces to the obvious but nevertheless important fact that all 2×2 skew-symmetric matrices form a commutative Lie algebra.

6. Returning to the general case we observed in §3.5 that a necessary condition for a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ to satisfy the standard identity of degree $2p$ was that $\varphi^*(\underline{b}_{2p+1}) = 0$ in case $\varphi: \mathfrak{k} \to \mathfrak{g}$ is the injection map. Here we shall give a necessary and sufficient condition in terms of the filtration of $C^*(\mathfrak{g})$ defined by the subalgebra $\mathfrak{k}$. We refer to the filtration $B_j \subset C^*(\mathfrak{g})$, $B_{j+1} \subset B_j$ used to define the spectral sequence for $H^*(\mathfrak{g})$ relative to $\mathfrak{k}$ (see [2], §14). That is, let F and E, $F \subset E$, be the underlying vector spaces for $\mathfrak{k}$ and $\mathfrak{g}$. Let $D \subset E^*$ be the orthogonal complement to F. We recall B_j is the ideal in $C^*(\mathfrak{g})$ generated by $\Lambda^j D$, $j = 0, 1, \cdots$. Thus if $r = \dim F$ we have

$$0 = B_{r+1} \subset B_r \subset \cdots \subset B_1 \subset B_0 \subset C^*(\mathfrak{g})$$

Here we shall be interested in the determination of which B_i contains the various invariant cocycles b_j.

Now since B_1 is the orthogonal complement to $\bigwedge F$, it is clear that B_1 is the kernel of $\varphi^*: C^*(\mathfrak{g}) \to C^*(\mathfrak{k})$. Hence since invariant cocycles go into invariant cocycles under φ^* and since a non-zero invariant cocycle is never cohomologous to zero one has $\varphi^*(b_{2p+1}) = 0$ if and only if $b_{2p+1} \in B_1$. We shall now see that it is the ideal B_2 which gives the standard identity.

Theorem 3.6. *Let $B_0 \supset B_1 \quad B_2 \cdots$ define the filtration of $C^*(\mathfrak{g})$ given by the subalgebra $\mathfrak{k}$. Then $\mathfrak{k}$ satisfies the standard identity of degree $2p$ if and only if $b_{2p+1} \in B_2$.*

Proof: If $b_{2p+1} \in B_2$ then clearly

$$\langle b_{2p+1}, B \wedge A_1 \wedge A_2 \wedge \cdots \wedge A_{2p} \rangle = 0 \qquad (3.6.1)$$

for all $A_1, A_2, \cdots, A_{2p} \in \mathfrak{k}$ and $B \in \mathfrak{g}$. But then by Lemma 3.4 this implies

$$\operatorname{tr} B \cdot [A_1 \cdots A_{2p}] = 0 \qquad (3.6.2)$$

for all $B \in \mathfrak{g}$ and hence $[A_1 A_2 \cdots A_{2p}] = 0$. Conversely, if $\mathfrak{k}$ satisfies the standard identity of degree $2p$ then, reversing the arguments, (3.6.2) holds for all $A_1, \cdots, A_{2p} \in \mathfrak{k}$ and $B \in \mathfrak{g}$. Now let M be the ideal in $C_*(\mathfrak{g})$ generated by $\bigwedge^{2p} F$. Let $M_{2p+1} = M \cap C_{2p+1}(\mathfrak{g})$. Clearly M_{2p+1} is linearly generated by all elements of the form $B \wedge A_1 \wedge \cdots \wedge A_{2p}$ where $B \in \mathfrak{g}$, $A_1, A_2, \cdots, A_{2p} \in \mathfrak{k}$.

Let $M^{2p+1} = M \subset C^{2p+1}(\mathfrak{g})$ be the orthocomplement of M_{2p+1} intersected with $C^{2p+1}(\mathfrak{g})$. It is straightforward to verify that $M^{2p+1} = B_2 \cap C^{2p+1}(\mathfrak{g})$. This can be done by picking a complement to $\mathfrak{k}$ in $\mathfrak{g}$ and then choosing a basis appropriately. But now (3.6.1) implies $b_{2p+1} \in M^{2p+1} \subset B_2$. Q.E.D.

4. The Frobenius theorem. We wish to put the theorem of Amitsur-Levitski in a different light. Here, first of all, no use will be made of modern cohomology theory. Instead, by means of a generalization of a result of Schur, [8], Hilfssatz III, we will show that the Amitsur-Levitski theorem is directly equivalent to a part of a remarkable theorem of Frobenius published in 1901.

The theorem of Frobenius is a theorem on the alternating group or more specifically on the characters of the alternating group $\mathcal{A} = \mathcal{A}_r$. By 1901 Frobenius has already obtained his famous formula for the characters of the symmetric group $\mathcal{S} = \mathcal{S}_r$. In [4] he turned his attention to the alternating group. The transition is by no means automatic and his work led to the discovery of a deep and beautiful property of $\mathcal{A}$. The original proof and the one in [3], as we have already remarked, are subtle ones and use ideas outside of group theory.

Involved in the statement and application (as given here) of the Frobenius theorem is the relationship between the representations of $\mathcal{S}$ and those of $\mathcal{A}$. We recall some of the general theory concerning this relationship; part

of this theory applies to the general case when one is dealing with a group and a normal subgroup of index 2 (see [3], §12, p. 86).

2. Consider an irreducible representation C of $\mathcal{S}$ in the complex vector space V (written here sometimes as $\{C,V\}$). Then if $D = C|\mathcal{a}$ is the restriction of this representation to $\mathcal{a}$, V may or may not be irreducible under D. Thus two cases arise. In case D is not irreducible it is easily seen that D may be written uniquely as the direct sum $D^{(1)} + D^{(-1)}$ where $D^{(1)}$ and $D^{(-1)}$ are inequivalent irreducible representations of $\mathcal{a}$. Furthermore, if the representations $D^{(1)}$ and $D^{(-1)}$ are given more precisely as $\{D^{(1)}, V^{(1)}\}$ and $\{D^{(1)}, V^{(-1)}\}$ where $V = V^{(1)} + V^{(-1)}$ is a direct sum then

$$C(\xi) : V^{(1)} \to V^{(-1)}$$

and (4.2.1)

$$C(\xi) : V^{(-1)} \to V^{(1)}$$

for any odd $\xi \in \mathcal{S}$.

It follows easily from (4.2.1) that $D^{(-1)}$ is equivalent to the representation $\sigma \to D^{(1)}(\xi\sigma\xi^{-1}), \sigma \in \mathcal{a}, \xi \notin \mathcal{a}$, of $\mathcal{a}$. In fact, this leads to a characterization of those irreducible representations (occurring in pairs) of $\mathcal{a}$ which are of the form $D^{(1)}$ or $D^{(-1)}$. That is, if D' is an irreducible representation of Λ, let D'' be given by $D''(\sigma) = D'(\xi\sigma\xi^{-1})$. Then if D' and D'' are inequivalent there exist irreducible representations $D^{(1)}$ and $D^{(-1)}$, arising as above from an irreducible representation of $\mathcal{S}$, which are

respectively equivalent to D' and D'' (see [3], pp. 86-91).

If $\chi^{(1)}$ and $\chi^{(-1)}$ are the characters of $D^{(1)}$ and $D^{(-1)}$ then it follows from the preceding that

$$\chi^{(-1)}(\sigma) = \chi^{(1)}(\xi\sigma\xi^{-1}) \qquad (4.2.2)$$

for every odd ξ and even σ. Of course, in case $D = C|\mathcal{A}$ is irreducible D is equivalent to the representation $\sigma \to D(\xi\sigma\xi^{-1})$.

Now let $\{C_\alpha, V_\alpha\}$, where α runs through the index set Ω, be a complete set of inequivalent irreducible representations of $\mathcal{S}$. It is a simple matter to show that the representations $D_\alpha = C_\alpha|\mathcal{A}$ of $\mathcal{A}$, if D_α is irreducible, and the representations $D_\alpha^{(1)}$ and $D_\alpha^{(-1)}$, if D_α is not irreducible, form a complete system (possibly with repetitions) of irreducible representations of $\mathcal{A}$. Now in case D_α is irreducible the associated character χ_α of $\mathcal{A}$ is, of course, obtained by restricting the known character of C_α to $\mathcal{A}$. In case D_α is reducible, however, then for a similar reason the characters $\chi_\alpha^{(1)}$ and $\chi_\alpha^{(-1)}$, respectively, are known as soon as one knows the difference

$$\chi_\alpha^{(1)} - \chi_\alpha^{(-1)}. \qquad (4.2.3)$$

The theorem of Frobenius gives this difference. One of its remarkable features is that as a function on $\mathcal{S}$ it has its support on only <u>one</u> conjugate class of $\mathcal{S}$. That is, there is associated a partition of r to each α for which D_α is not reducible. The partition is one with odd

distinct parts and the correspondence thus set up between all such α and all such partitions is one-one. [The flavor of cohomology (that is, of the $n \times n$ unitary group) first becomes apparent here when it is realized that the r^{th} Betti number of the unitary group is just the number of all such partitions. The precise relationship between these functions and a basis of $H^r(\mathcal{G})$ will be made in §5.] We shall find it convenient to use the more conventional Young tableau theory rather than that of Frobenius to describe these matters.

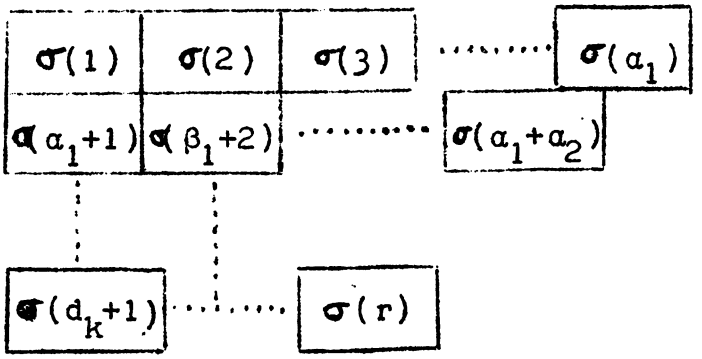
3. Let I be the complex group algebra of $\mathcal{S} = \mathcal{S}_r$. An element $f \in I$ will be regarded both as a function on $\mathcal{S}$ and as a complex linear combination of its elements. That is, if as the latter $f = \sum_{\sigma \in \mathcal{S}} a_\sigma \sigma$ then as the former $f(\sigma) = a_\sigma$.

As one knows, I can be uniquely written $I = \sum_{\alpha \in \Omega} I_\alpha$ where the I_α, $\alpha \in \Omega$, are the minimal 2-sided ideals in I. The index set Ω is the same as that of §4.2 where the kernel of C_α is the maximal ideal $\sum_{\beta \neq \alpha} I_\beta$ and C_α maps I_α isomorphically onto $\mathcal{G}(V_\alpha)$, the full linear algebra on V_α.

We recall that the set Ω is identified with the set of all partitions of r as follows: For each partition $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$, $\sum_{i=1}^{k} \alpha_i = r$ we construct an associated graph $G(\alpha)$ by placing α_1 nodes in the first row, α_2 nodes in the second row, $\cdots$, α_k nodes in the k^{th} row. This is done in such a way that

the j^{th} node in each row lies in a vertical column, the j^{th} column. If $j \leq \alpha_i$ then there is a node in the i^{th} row, j^{th} column. In such a case we will let (i,j) designate the node and we will write $(i,j) \in G(\alpha)$. Also given α we will always let $d_i = d_i(\alpha) = \sum_{j=1}^{i-1} \alpha_j$.

Now associated to every partition $\alpha \in \Omega$ and every permutation $\sigma \in \mathcal{S}$ is the corresponding Young tableau or diagram $Y(\alpha:\sigma)$. One obtains $Y(\alpha:\tau)$ by associating to each node of $G(\alpha)$ a number from 1 to r (this being done in a one-one way) as follows: If $(i,j) \in G(\alpha)$ then the associated number is $\sigma(d_{i+j})$. The Young diagram $Y(\alpha:\sigma)$ is usually realized by the following configuration



Now let $\mathcal{R}(\alpha:\sigma)$ (resp. $\mathcal{C}(\alpha:\sigma)$) be the subgroup of $\mathcal{S}$ consisting of all elements of $\mathcal{S}$ which leave the rows (resp. the columns) of $Y(\alpha:\sigma)$ invariant.

Let

$$P(\alpha:\sigma) = \sum_{\rho \in \mathcal{C}(\alpha:\sigma)}$$

and

$$N(\alpha:\sigma) = \sum_{\tau \in \mathcal{R}(\alpha:\sigma)} sg(\tau)\,\tau.$$

Then if $\mathcal{O}(\mathcal{R})$ and $\mathcal{O}(\mathcal{C})$ are respectively the orders of

$\mathcal{R}(\alpha:\sigma)$ and $\mathcal{C}(\alpha:\sigma)$ the elements $P_o = P_o(\alpha:\sigma)$ and $N_o = N_o(\alpha:\sigma)$ defined by $P_o = [1/o(\mathcal{R})]P$ and $N_o = [1/o(\mathcal{C})]N$ are idempotents in I. Here $P = P(\alpha:\sigma)$ and $N = N(\alpha:\sigma)$. One defines the Young symmetrizer $E = E(\alpha:\sigma)$ by letting

$$E = PN. \tag{4.3.1}$$

Then, apart from a scalar multiple, E is also an idempotent in I. But more than that, the idempotent in this case is minimal so that E, in addition, belongs to one of the minimal 2-sided ideals in I. That this ideal depends only on α and not on σ is a consequence of the fact that

$$E(\alpha:\sigma_1) = \sigma E(\alpha:\sigma_2)\sigma^{-1}$$

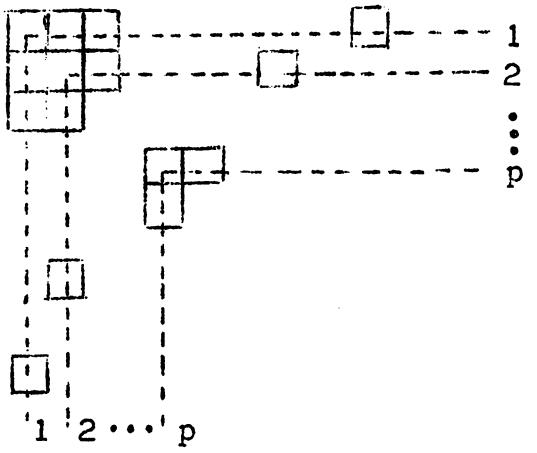
for $\sigma = \sigma_1\sigma_2^{-1}$. The correspondence thus set up between partitions α and minimal 2-sided ideals is one-one onto and hence establishes the use of the index set Ω for the minimal 2-sided ideals and the corresponding irreducible representations of $\mathcal{S}$ or I (see §4.2).

4. Now important for our considerations is the notion of partition α and associated partition α^t.

The partition α^t is best defined by giving its graph $G(\alpha^t)$. The latter is just the graph obtained from $G(\alpha)$ by transposing as one would a matrix. Thus $(i,j) \in G(\alpha^t)$ if and only if $(j,i) \in G(\alpha)$.

5. Now to each $\beta \in \Omega_o$ there corresponds by what we shall call the "shell mapping" another partition $\delta = \delta(\beta)$ consisting of distinct odd parts. Defined in an algebraic way (seemingly uninteresting) $\delta = (\delta_1, \delta_2, \cdots, \delta_p)$ is given by $\delta_j = 2(\beta_j - j) - 1$ where p is the highest number

for which $(p,p) \in G(\beta)$. On the other hand, obtaining δ directly from the graph $G(\beta)$ amounts to letting δ_q be the number of nodes in the q^{th} shell of $G(\beta)$. Here (i,j) is in the q^{th} shell if either i or j equals q. In the graph $G(\beta)$ below numbered lines are run through the corresponding shells. The first and necessarily longest shell we call the outer shell.



Let $\Delta \subset \Omega$ be the set of all partitions with distinct odd parts. The shell mapping

$$\delta: \Omega_o \to \Delta$$

is clearly one-one onto.

Let $\{C_\alpha, V_\alpha\}$, $\alpha \in \Omega$, be as in §4.2. We can now be more definite as to when $D_\alpha = C_\alpha|A$ remains irreducible or decomposes (see §4.2). The following theorem is easy to prove (see §6, p. 183, [3]).

Theorem 4.5. Let the representations $\{C_\alpha, V_\alpha\}$ $\alpha \in \Omega$, of the symmetric group $\mathcal{S}$ be as in §4.2. Let $D_\alpha = C_\alpha|\mathcal{A}$ be the restriction to the alternating group $\mathcal{A}$. Then D_α is reducible if and only if $\alpha = \beta \in \Omega_o$. (See §4.5.)

6. Now let $\beta \in \Omega_0$ and let $D_\beta^{(1)}$ and $D_\beta^{(-1)}$ be the two irreducible components of $D_\beta = C_\beta | \mathcal{A}$. (See §4.2.) If $\chi_\beta^{(1)}$ and $\chi_\beta^{(-1)}$ are the corresponding characters then (see (4.2.2))

$$\chi_\beta^{(-1)}(\sigma) = \chi_\beta^{(1)}(\xi \sigma \xi^{-1})$$

for any $\sigma \in \mathcal{A}$, $\xi \notin \mathcal{A}$. Hence $\chi_\beta^{(1)}$ and $\chi_\beta^{(-1)}$, as class functions on $\mathcal{A}$, can differ only on those conjugate classes $\mathcal{F}_0$ of $\mathcal{A}$ which are not conjugate classes of $\mathcal{S}$.

Now for any $\delta \in \Omega$ let $\mathcal{F}_\delta \subset \mathcal{S}$ be the conjugate class of $\mathcal{S}$ whose cycle structure is given by δ. Now either $\mathcal{F}_\delta \subset \mathcal{A}$ or $\mathcal{F}_\delta \cap \mathcal{A}$ is empty. If $\mathcal{F}_\delta \subset \mathcal{A}$ then either $\mathcal{F}_\delta$ remains a conjugate class of $\mathcal{A}$ or it decomposes into two classes $\mathcal{F}_\delta^{(+)}$ and $\mathcal{F}_\delta^{(-)}$ where $\xi \mathcal{F}_\delta^{(+)} \xi^{-1} = \mathcal{F}_\delta^{(-)}$ for any odd $\xi \in \mathcal{S}$. It is a simple matter to see that this decomposition takes place if and only if the subgroup $\mathcal{S}(\sigma)$ of all permutations commuting with any $\sigma \in \mathcal{F}_\delta$ is contained wholly in $\mathcal{A}$. It is elementary then to verify that this decomposition takes place if and only if $\delta \in \Delta$.

Thus if $\beta \in \Omega_0$, $\chi_\beta^{(1)}$ and $\chi_\beta^{(-1)}$ can differ on $\sigma \in \mathcal{A}$ only if $\sigma \in \mathcal{F}_\delta$ where $\delta \in \Delta$. The theorem of Frobenius asserts that there is exactly one δ for which a difference occurs and that δ is just $\mathcal{S}(\beta)$.

Theorem 4.6. (Frobenius). Let $\chi_\beta^{(1)}$ and $\chi_\beta^{(-1)}$ be the characters of the alternating group $\mathcal{A} = \mathcal{A}_r$ arising from a self-associated partition β of r (see §4.5). Let $\delta = \mathcal{S}(\beta)$ be the partition of r with distinct odd parts obtained from β by the "shell mapping" (see §4.5). Then, as functions

on $\mathcal{A}$, $\chi_\beta^{(1)}$ and $\chi_\beta^{(-1)}$ are identical on every even permutation whose cycle structure is not $\delta(\sigma \notin \mathcal{F}_\delta)$. On the other hand, if $\sigma \in \mathcal{F}_\delta^{(+)}$ and $\tau \in \mathcal{F}_\delta^{(-1)}$ then the sign (appearing in the superscript of χ_β) may be chosen so that

$$\chi_\beta^{(1)}(\sigma) = \tfrac{1}{2}(\mu + \sqrt{\mu s}); \quad \chi_\beta^{(1)}(\tau) = \tfrac{1}{2}(\mu - \sqrt{\mu s}),$$

$$\chi_\beta^{(-1)}(\sigma) = \tfrac{1}{2}(\mu - \sqrt{\mu s}); \quad \chi_\beta^{(-1)}(\tau) = \tfrac{1}{2}(\mu + \sqrt{\mu s})$$

where $s = \delta_1 \delta_2 \cdots \delta_p$ and $\mu = (-1)^{\frac{1}{2}(s-1)}$.

Now for $\sigma \in \mathcal{S}$ let $h^\sigma \in I$ be defined by

$$h^\sigma = \frac{1}{r!} \sum_{\rho \in \mathcal{S}} sg(\rho) \rho \sigma \rho^{-1} . \qquad (4.6.1)$$

It is immediate that $h^\sigma \neq 0$ if and only if the centralizer $\mathcal{S}(\sigma)$ of σ is contained in $\mathcal{A}$. That is, $h^\sigma \neq 0$ if and only if $\sigma \in \mathcal{F}_\delta$ where $\delta \in \Delta$. Furthermore, $h^\sigma = -h^\tau$ if $\sigma \in \mathcal{F}_\delta^{(+)}$ and $\tau \in \mathcal{F}_\delta^{(-1)}$. To normalize the choice of sign in $\mathcal{F}_\delta^{(+)}$ and $\mathcal{F}_\delta^{(-1)}$ when $\delta \in \Delta$ let $\mathcal{F}_\delta^{(+)}$ be the conjugate class of $\mathcal{A}$ which contains the element

$$\sigma = (1,2,\cdots,\delta_1)(\delta_1+1,\delta_1+2,\cdots,\delta_1+\delta_2)\cdots(d_p+1,d_p+2,\cdots,n)$$

where $d_p = \sum_{i=1}^{p-1} \delta_i$.

Now if $\delta \in \Delta$ and $\mathcal{S}(\beta) = \delta$ define

$$h_\beta = \sum_{\sigma \in \mathcal{F}_\delta^{(+)}} \sigma - \sum_{\tau \in \mathcal{F}_\delta^{(-)}} \tau. \qquad (4.6.2)$$

Clearly there exists a positive scalar c such that $h_\beta = ch^\sigma$ in case $\sigma \in \mathcal{F}_{\mathcal{S}(\beta)}^{(+)}$. On the other hand, observe that the

Frobenius theorem asserts

$$\chi_\beta^{(1)} - \chi_\beta^{(-1)} = \sqrt{\mu}\, sh_\beta \qquad (4.6.3)$$

where μ and s are defined in Theorem 4.6 and $\delta = \delta(\beta)$.

Now let Γ: I → I be the operator defined by

$$\Gamma(f) = \frac{1}{r!} \sum_{\rho \in \mathcal{S}} sg(\rho) \rho f \rho^{-1}. \qquad (4.6.4)$$

Then $\Gamma^2 = \Gamma$ and if H $\subset$ I is the range of Γ then H is the subspace of all elements of I which commute with even permutations and anti-commute with odd permutations. Now if $f \in I$ is written $f = \sum_{\sigma \in \mathcal{S}} a_\sigma \sigma$ then clearly

$$\Gamma(f) = \sum_{\tau \in \mathcal{S}} a_\sigma h^\sigma.$$

Thus the elements h^σ, $\sigma \in \mathcal{S}$, span H; or, from above, the elements h_β, $\beta \in \Omega_o$, span H. On the other hand, the elements h_β, $\beta \in \Omega_o$, are linearly independent in I. This is obvious since as functions on $\mathcal{S}$ they have their supports on disjoint sets (the conjugate clases F_δ, $\delta \in \Delta$). Thus the dimension of H is equal to the number of partitions of r with distinct odd parts.

Now for any $\alpha \in \Omega$ let e_α be the identity element of the simple ideal I_α. Now if $\beta \in \Omega_o$ we observe that $\chi_\beta^{(1)} + \chi_\beta^{(-1)}$ is the character of C_β as well as D_β. This is clear since the character of C_β vanishes on odd ξ. Indeed, this is a consequence of (4.2.1). The latter clearly implies tr $C_\beta(\xi) = 0$. Thus

$$e_\beta = \frac{c}{r!} (\chi_\beta^{(1)} + \chi_\beta^{(-1)})$$

where $c = \dim V_\beta$. But then $e_\beta \cdot \chi_\beta^{(1)} = \chi_\beta^{(1)} \in I_\beta$ and $e_\beta \cdot \chi_\beta^{(-1)} = \chi_\beta^{(-1)} \in I_\beta$. This, together with (4.6.3), implies the following corollary. Corollary 4.6 is, for us, the main point here. All the results of this section will be deduced from it.

Corollary 4.6. Let the notation be as in §§4.5 and 4.6. Then for every $\beta \in \Omega_o$

$$h_\beta \in I_\beta.$$

Actually it is not difficult to show that this corollary is an equivalent formulation of the Frobenius theorem. The particular constants occurring in Theorem 4.6 are easily derived once one knows $h_\beta \in I_\beta$.

An immediate consequence of the fact that $h_\beta \in I_\beta$ for all $\beta \in \Omega_o$ is that $h^\sigma \cdot h^\tau = 0$ whenever σ and τ are not conjugate in $\mathcal{S}$. This simple fact seems to be difficult to prove directly.

7. Since we shall be concerned later with proving equivalences we wish to record here certain facts about H which can be obtained without the use of the Frobenius theorem.

For one thing, the statement that the elements $\{h_\beta\}$, $\beta \in \Omega_o$, form a basis of H, of course, does not depend upon Theorem 4.6. Next it is clear that if we define h_β^o by

$$h_\beta^o = \frac{1}{\sqrt{\mu_s}} (\chi_\beta^{(1)} - \chi_\beta^{(-1)})$$

then $h_\beta^o \in H$ and that the set $\{h_\beta^o\}$, $\beta \in \Omega_o$, is likewise a

basis. That is, they are obviously independent and since there are enough of them they, too, form a basis. Perhaps a more interesting argument for this is as follows. Note that the characterization of the space H given in §4.6 implies $C_\alpha(H) = 0$ for all $\alpha \notin \Omega_o$. (Hence $H_\alpha = H \cap I_\alpha = 0$.) Indeed, this is a consequence of the irreducibility of D_α. On the other hand, it is just as clear that $C_\beta(H)$ is one-dimensional for every $\beta \in \Omega_o$ (and hence $H_\beta = H \cap I_\beta$ is one-dimensional). Indeed, an operator on V_β commutes with $C_\beta(\sigma)$ for even σ and anti-commutes with $C_\beta(\sigma)$ for odd σ if and only if it acts as a scalar on $V_\beta^{(1)}$ and minus that scalar on $V_\beta^{(-1)}$. Clearly $C_\beta(h_\beta^o)$ satisfies this condition. Since, obviously, $H = \sum_{\alpha \in \Omega} H_\alpha$ it follows that the h_β^o form a basis of H. Thus without the knowledge $h_\beta = h_\beta^o \in I_\beta$ we have at least

$$h_{\beta'} \in \sum_{\beta \in \Omega_o} I_\beta \qquad (4.7.1)$$

for all $\beta' \in \Omega_o$.

8. Let U be an m-dimensional complex vector space. Let ${}^rU$ designate the r-fold tensor product $\bigotimes^r U$ of U.

Let B designate the representation of $\mathcal{S} = \mathcal{S}_r$ on ${}^rU$ given by

$$B(\sigma)x_1 \otimes x_2 \otimes \cdots \otimes x_r = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(r)}$$

where $x_i \in U$, $i = 1, 2, \cdots, r$.

Now ${}^rU$ decomposes uniquely into the direct sum

$${}^rU = \sum_{\alpha \in \Omega} U_\alpha \qquad (4.8.1)$$

where U_α is the subspace of all tensors transforming under the action of $\mathcal{S}$ according to the representation C_α, $\alpha \in \Omega$ (quantics of type α). The subspace U_α is, of course, simply the range of the projection $B(e_\alpha)$. One necessarily has

$$B(f_{\alpha_1})t_{\alpha_2} = 0$$

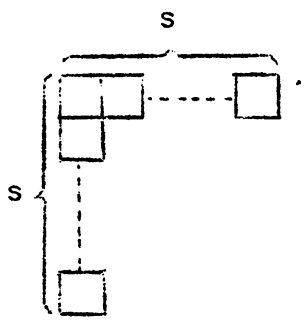
if $f_{\alpha_1} \in I_{\alpha_1}$, $t_{\alpha_2} \in U_{\alpha_2}$ and $\alpha_1 \neq \alpha_2$.

Now the subspace U_α, for some $\alpha \in \Omega$, may very well reduce to 0. This, of course, depends upon the dimension n of U. Unless C_α is the identity representation this will happen if n is sufficiently small. The precise condition (Theorem 4.4C, [5], p. 129) is well known and is given below. The proof is easy and may be obtained as a simple consequence of three facts: (1) e_α is a scalar multiple of the sum $\sum_\sigma E(\alpha:\sigma)$ of all r! Young symmetrizers in I_α, (2) $E(\alpha:\sigma) = E = PN$ (see 4.3.1)) and (3) $B(N)t \neq 0$ for the decomposable tensor $t = x_1 \otimes x_2 \otimes \cdots \otimes x_r$ only when there are sufficiently many linearly independent x_i.

For any $\alpha \in \Omega$ let $\ell(\alpha)$ be the number of parts in the partition α.

Lemma 4.8. *Let $\alpha \in \Omega$; then $U_\alpha \neq 0$ if and only if $\ell(\alpha) \leq n$.*

Now assume r is odd, say $r = 2s - 1$. Consider the self-associated partition $\epsilon = \epsilon(r) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_s)$ defined by $\epsilon_1 = s$ and $\epsilon_2 = \epsilon_3 = \cdots = \epsilon_s = 1$. The graph $G(\epsilon)$ is



(4.8.2)

The partition $\epsilon \in \Omega_o$ is unique in that its graph has only one shell and hence the number of its rows and columns are maximum.

Let $\gamma = S(\epsilon)$. Then of course $\gamma = (\gamma_1) \in \Delta$ is the partition of r with only one part; $\gamma_1 = r$.

Now if we apply the Frobenius theorem, or rather Corollary 4.6, to the case where $\beta = \epsilon$ we have $h_\epsilon \in I_\epsilon$. We now observe that this part of the Frobenius theorem is equivalent to the statement that $B(h_\epsilon) = 0$ if and only if $s > n$. This is an immediate consequence of Lemma 4.8 and (4.7.2). Hence the Frobenius theorem implies and is implied by

Theorem 4.8. *Let* $r = 2s - 1$ *and let* $\sigma \in \mathcal{F}_\gamma$ *be the cycle* $(1, 2, \cdots, r)$. *Then if* B *is the representation of* $\mathcal{S}_r$ *on* ${}^rU$ *given in* §4.8,

$$B(\sum_{\tau \in \mathcal{S}} sg(\tau)\tau\sigma\tau^{-1}) = 0$$

if and only if $s > n$.

9. Let $\mathcal{g} = \mathcal{g}(U)$ be the full linear algebra of operators on U and ${}^r\mathcal{g} = \mathcal{g}({}^rU)$ the full linear algebra of operators on ${}^rU$. The notation ${}^r\mathcal{g}$ is consistent with that of §4.8 since, as one knows, ${}^r\mathcal{g}$ may be identified with $\otimes^r\mathcal{g}$

where in fact one has

$$(A_1 \otimes A_2 \otimes \cdots \otimes A_r)x_1 \otimes x_2 \otimes \cdots \otimes x_r =$$

$$A_1x_1 \otimes A_2x_2 \otimes \cdots \otimes A_rx_r$$

for $A_i \in \mathcal{g}$; $x_j \in U$.

Now, as observed in §3.5, one has an immediate equivalence between the Amitsur-Levitski theorem and (3.4.3). We now show that the Amitsur-Levitski theorem is equivalent to a part of the Frobenius theorem ($h_\epsilon \in I_\epsilon$) by showing the equivalence of Theorems 4.8 and (3.4.3). This equivalence will be an immediate consequence of the following generalization of a result of I. Schur. Lemma 4.9 reduces to the result of Schur when $A_1 = A_2 = \cdots = A_r$.

Lemma 4.9. *Let $\rho \in \mathfrak{S}_r$ and let the cycle decomposition of ρ^{-1} be given by*

$$\rho^{-1} = (i_1 \cdots i_{\delta_1})(i_{\delta_1+1} \cdots i_{\delta_1+\delta_2}) \cdots (i_{d_{k+1}} \cdots i_r)$$

where $i_1, i_2, \cdots, i_r$ is a permutation of $1, 2, \cdots, r$ and $d_k = \sum_{i=1}^{k-1} \delta_i$. Let $A_1, A_2, \cdots, A_r \in \mathcal{g} = \mathcal{g}(U)$. Then

$$\operatorname{tr} B(\rho) \cdot A_1 \otimes A_2 \otimes \cdots \otimes A_r \qquad (4.9.1)$$

$$= \operatorname{tr}(A_{i_1} \cdots A_{i\delta_1}) \operatorname{tr}(A_{i\delta_1+1} \cdots A_{i\delta_1+\delta_2}) \cdots \operatorname{tr}(A_{i_{d_k+1}} \cdots A_{i_r}).$$

(It should be noted that the trace taken on the right side of this equation is for operators on U.)

Proof: Let U^* be the dual space to U. To prove the lemma it sufficies by linearity to prove it when all the A_i are of the form $x \otimes \omega$. Here $x \in U$ and $\omega \in U^*$ and $x \otimes \omega$ is that operator on U defined by $x \otimes \omega(y) = \langle \omega, y \rangle x$ for any $y \in U$. Note that $x_1 \otimes \omega_1 \cdot x_2 \otimes \omega_2 = \langle \omega_1, x_2 \rangle x_1 \otimes \omega_2$ and $\operatorname{tr} x \otimes \omega = \langle x, \omega \rangle$.

Now ${}^r(U^*)$ is naturally identified with $({}^rU)^*$ (and hence can be written ${}^rU^*$) where

$$\langle x_1 \otimes \cdots \otimes x_r, \omega_1 \otimes \cdots \otimes \omega_r \rangle = \prod_{j=1}^{r} \langle x_i, \omega_i \rangle .$$

Let $e_j \in U$, $j = 1, 2, \cdots, n$, be a basis of U and let $\theta_j \in U^*$ be the dual basis. Now if C is any operator on ${}^rU$ then

$$\operatorname{tr} C = \sum_{1 \leq j_i \leq n} \langle C(e_{j_1} \otimes \cdots \otimes e_{j_n}), \theta_{j_1} \otimes \cdots \otimes \theta_{j_n} \rangle$$

Now assume $A_i = x_i \otimes \omega_i$, $i = 1, 2, \cdots, r$. Then if $C = A_1 \otimes A_2 \otimes \cdots \otimes A_r \cdot B(\rho)$

$$\langle C(e_{j_1} \otimes \cdots \otimes e_{j_r}), \theta_{j_1} \otimes \cdots \otimes \theta_{j_r} \rangle = \prod_{i=1}^{r} \langle x_i, \theta_{j_i} \rangle \langle e_{j_{\rho^{-1}(i)}}, \omega_i \rangle$$

$$= \prod_{i=1}^{r} \langle x_i, \theta_{j_i} \rangle \langle e_{j_i}, \omega_{\rho(i)} \rangle .$$

But now if we sum this last expression (m^r terms) over all $1 \leq j_i \leq m$ it follows that

$$\operatorname{tr} C = \prod_{i=1}^{r} \langle x_i, \omega_{\rho(i)} \rangle . \qquad (4.9.2)$$

Now to prove the lemma it suffices only to rearrange the the factors in (4.9.2) so that it takes the form of (4.9.1). Indeed, (4.9.2) may be rearranged so that

$$\text{tr } C = \left(\langle x_{i_1}, \omega_{i\,\delta_1}\rangle \prod_{p=2}^{\delta_1} \langle x_{i_p}, \omega_{i_{p-1}}\rangle\right)$$

$$\left(\langle x_{i\,\delta_1+1}, \omega_{i\,\delta_1+\delta_2} \prod_{p=2}^{\delta_2} \langle x_{i\,\delta_1+p}, \omega_{i\,\delta_1+p-1}\rangle\right)$$

$$\cdots\left(\langle x_{i_{d_{k+1}}}, \omega_{i_r}\rangle \prod_{p=2}^{\delta_k} \langle x_{i_{d_{k+p}}}, \omega_{i_{d_{k+p-1}}}\rangle\right).$$

But now from the first paragraph in this proof the terms in the parenthesis are respectively

$$(\text{tr } A_i \cdots A_{i\,\delta_1}),\ (\text{tr } A_{i\,\delta_1+1} \cdots A_{i\,\delta_1+\delta_2}), \cdots, (\text{tr } A_{i_{d_{k+1}}} \cdots A_{i_r}).$$

Q.E.D.

Now in Lemma 4.9 let $\rho = \tau\sigma\tau^{-1}$ where

$$\sigma^{-1} = (1, 2, \cdots, r).$$

It follows then from Lemma 4.9 that

$$\text{tr } B(\tau\sigma\tau^{-1}) \cdot A_1 \otimes A_2 \otimes \cdots \otimes A_r = \text{tr } A_{\tau(1)} A_{\tau(2)} \cdots A_{\tau(r)}$$

and hence

$$\text{tr } B\left(\sum_{\tau \in S} \text{sg}(\tau)\, \tau\sigma\tau^{-1}\right) \cdot A_1 \otimes A_2 \otimes \cdots \otimes A_r =$$

$$\text{tr } [A_1 A_2 \cdots A_r]. \qquad (4.9.3)$$

We can now go back and forth between Theorem 4.8 and (3.4.3). That is, (4.9.3) implies $B(h_\epsilon) = 0$ if and only if $tr(A_1A_2 \cdots A_r] = 0$ for all $A_1, A_2, \cdots, A_r \in \mathfrak{g}$. Setting $s = n + 1$ so that $r = 2n + 1$, Theorem 4.8 implies

$$tr[A_1A_2 \cdots A_{2n+1}] = 0$$

for all $A_1, A_2, \cdots, A_{2n+1} \in \mathfrak{g}$ and conversely.

5. Cohomology classes as operators. 1. In this section we complete the circle of ideas relating the Amitsur-Levitski theorem, the theorem of Frobenius and the formula of Dynkin. We return to the cohomology theory of Lie algebras specializing to the case where $\mathfrak{g} = \mathfrak{g}(U)$, the Lie algebra of all operators on U. First, by use of part of the Frobenius theorem ($h_\epsilon \in I_\epsilon$) we shall exhibit a natural relationship between the primitive r-cohomology classes of $\mathfrak{g}$ (r odd) and the operator $B(h_\epsilon)$. As a consequence of this we obtain a proof of the formula of Dynkin.[3] This then completes the proof of the equivalence of I, II and III of §1.2. Also as a consequence we sharpen a result of Weyl. (See 5.5 here.) Most important (in the author's opinion), it is demonstrated that the space of tensors U_ϵ acts as a focal point where the three theorems of §1.2 come together. Finally the circle of dependencies among these theorems is broken into by giving a direct proof of the formula of Dynkin. This proof is based on the operator-theoretic

[3] We wish to observe that in [7]. p. 141, Dynkin isolates a particular class in $P^r(\mathfrak{g})$. Since our concern is with cohomology over fields, no such distinction is made here.

significance of cup and Pontrjagin products.

2. Let U and ${}^rU$, $\mathcal{J} = \mathcal{J}(U)$ and ${}^r\mathcal{J}$ be as in §4.8. Also, as in §4.8, let B designate the representation of $\mathcal{S} = \mathcal{S}_r$ on ${}^rU$.

Now let B^o be the representation of $\mathcal{S}$ on ${}^r\mathcal{J}$ obtained in the same way as B except that $\mathcal{J}$ is substituted for U. We will be especially interested in the subspace $\mathcal{J}_\omega \subset {}^r\mathcal{J}$ of anti-symmetric r-tensors over $\mathcal{J}$. That is, $\mathcal{J}_\omega$ is the range of the projection $B^o(e_\omega)$ where

$$e_\omega = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}} sg(\sigma)\sigma$$

is the identity element of the one-dimensional ideal I_ω. Here $\omega = \omega(r) = (\omega_1, \omega_2, \cdots, \omega_r) \in \Omega$ is the partition given by $\omega_1 = \omega_2 = \cdots \omega_r = 1$.

Now recall that ${}^r\mathcal{J}$ is also the full algebra of operators on ${}^rU$. It is easily verified that the representations B and B^o of $\mathcal{S}$ are related in the following way. For any $C \in {}^r\mathcal{J}$

$$B^o(\sigma)(C) = B(\sigma)CB(\sigma)^{-1}.$$

Indeed it suffices only to verify this for all C of the form $A_1 \otimes A_2 \otimes \cdots \otimes A_r$, $A_i \in \mathcal{J}$. In such a case it is obvious. Now then

$$B^o(e_\omega)C = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}} sg(\sigma)B(\sigma)CB(\sigma^{-1})$$

and hence the space $\mathcal{J}_\omega$ of anti-symmetric elements in ${}^r\mathcal{J}$ as operators on ${}^rU$ can be characterized as the set of all operators on ${}^rU$ which commute with all $B(\sigma)$ for even σ and anti-commute with all $B(\sigma)$ for odd σ.

Now consider the mapping

$$\eta_r : C_r(\mathfrak{g}) \to \mathfrak{g}_\omega(r)$$

given by

$$\eta_r(A_1 \otimes \cdots \otimes A_r) = B^0(e_\omega)(A_1 \otimes \cdots \otimes A_r).$$

Let the non-singular symmetric bilinear form (C_1, C_2) on ${}^r\mathfrak{g}$ be defined by

$$(C_1, C_2) = \operatorname{tr} C_1 C_2 \qquad (5.2.1)$$

It is clear that $B^0(\mathfrak{r})$ is orthogonal with respect to this inner product. This implies that $B^0(e_\omega)$ is symmetric with respect to it and consequently the bilinear form (5.2.1) is necessarily non-singular on $\mathfrak{g}_\omega$. We define

$$\eta^r : \mathfrak{g}_\omega \to C^r(\mathfrak{g})$$

by

$$\langle \eta^r(C), u \rangle = (C, \eta_r(u))$$

where $C \in \mathfrak{g}_\omega$ and $u \in C_r(\mathfrak{g})$. It is clear that both η_r and η^r are onto isomorphisms.

If $B_i, A_j \in \mathfrak{g}$, $i, j = 1, 2, \cdots, r$, then it is a simple fact that

$$(A_1 \otimes \cdots \otimes A_r,\ B_1 \otimes \cdots \otimes B_r) = \prod_{i=1}^{r} \operatorname{tr} A_i B_i .$$

(Also note that Lemma 4.9 reduces to this when $\rho = e$.) It follows then that if $\eta^r(A_i) = a_i \in C^r(\mathfrak{g})$,

$$\eta^r(B^0(e_\omega)(A_1 \otimes \cdots \otimes A_r)) = \frac{1}{r!}\, a_1 \wedge \cdots \wedge a_r . \qquad (5.2.2)$$

3. Let $G = G(U)$ and ${}^rG = G({}^rU)$ be the groups of non-singular operators, respectively, on U and ${}^rU$. Let $\pi : G \to {}^rG$

be the representation of G on ${}^rU$ defined by

$$\pi(A) = \underbrace{A \otimes \cdots \otimes A}_{r}.$$

Now the representation π of G on ${}^rU$ induces the representation π^0 of G on ${}^r\mathfrak{g}$ defined by

$$\pi^0(A)(C) = \pi(A)C\pi(A^{-1}) \qquad (5.3.1)$$

for $C \in {}^r\mathfrak{g}$. Since all $\pi(A)$ commute with all $B(\sigma)$ it follows trivially that $\mathfrak{g}_\omega$ is invariant under all $\pi^0(A)$. Let $\pi^0_\omega(A)$ be the restriction of $\pi^0(A)$ to $\mathfrak{g}_\omega$. Recall now the representation θ_r of $\mathfrak{g}$ on $C_r(\mathfrak{g})$ considered in §2.3. We now observe that the isomorphism η_r defines an equivalence between the representation θ_r and the differential representation of $\mathfrak{g}$ defined by π^0_ω. It follows then that η_r maps $J_r(\mathfrak{g})$ onto the invariants of π^0_ω. But it follows from (5.3.1) that the space of invariants of π^0_ω is exactly the set of all operators on $\mathfrak{g}_\omega$ which commute with all $\pi(A)$. We now apply the already classical result asserting that the set of all operators on ${}^rU$ which commute with all $\pi(A)$ is the set $B(I)$ (see[5], Theorems 3.5.D and 4.4.D). Thus if ψ_r is the restriction of η_r to $J^r(\mathfrak{g})$ it follows that the range of ψ_r is $\mathfrak{g}_\omega \cap B(I)$. On the other hand, by the definition of H we see that $\mathfrak{g}_\omega \cap B(I) = B(H)$. Furthermore, since $\pi^0_\omega(A)$ is orthogonal with respect to the bilinear form (5.2.1) it follows that this bilinear form is non-singular on $B(H)$. Hence if ψ^r is the restriction of η^r to $B(H)$ it follows that ψ^r maps $B(H)$ isomorphically onto $J^r(\mathfrak{g})$. Summarizing we have proved

Theorem 5.3. Let $\mathfrak{g}_{\omega}(r)$ be the space of anti-symmetric r-tensors over the Lie algebra of all operators $\mathfrak{g}$ on U. Let

$$\eta_r : C_r(\mathfrak{g}) \to \mathfrak{g}_{\omega(r)}$$

$$\eta^r : \mathfrak{g}_{\omega(r)} \to C^r(\mathfrak{g})$$

be the maps defined as in §5.2. Then if (1) $H = H_r$ is the subspace of the group algebra $I = I_r$ of the symmetric group $S = S_r$ defined as in §4.6, (2) B is the representation of I on the space ${}^rU$ of r-tensors over U given in §4.8, (3) $J_r(\mathfrak{g}) \subset C_r(\mathfrak{g})$ and $J^r(\mathfrak{g}) \subset C^r(\mathfrak{g})$ are, as in 5.3, respectively, the spaces of invariant r-cycles and r-cocycles and, finally, (4) ψ_r and ψ^r are, respectively, the restrictions of η_r to J_r and η^r to B(H) we have

$$\psi_r : J_r(\ ) \to B(H)$$

$$\psi^r : B(H) \to J^r(\mathfrak{g})$$

and, furthermore, ψ_r and ψ^r are both onto isomorphisms.

The shift to homology and cohomology is now immediate. Since $J_r(\mathfrak{g})$ and $J^r(\mathfrak{g})$ are respectively isomorphic to $H_r(\mathfrak{g})$ and $H^r(\mathfrak{g})$ under the mappings $u \to \underline{u}$ and $a \to \underline{a}$, ψ_r and ψ^r induce onto isomorphisms

$$\psi_r : H_r(\ ) \to B(H)$$

$$\psi^r : B(H) \to H^r(\mathfrak{g}).$$

The maps ψ_r and ψ^r neatly convert the study of the more or less intangible spaces $H_r(\mathfrak{g})$, $H^r(\mathfrak{g})$ to a study of the very

concrete space B(H). This is particularly significant in the light of the Frobenius theorem which gives an almost complete picture as to how B(H) acts as operators on ^{r}U, or more particularly on the subspaces U_β of tensors (quantics of type β) corresponding to the self-associated partitions β of r. We recall that the elements $\{h_\beta\}$, $\beta \in \Omega_0$, form a basis of H and that Corollary 4.6 implies that $B(h_\beta)$, which leaves U_β invariant, vanishes on all U_α, where $\alpha \neq \beta$. That this is especially relevant for the traditional study of either $H_*(\mathcal{G})$ (or $H^*(\mathcal{G})$) as an exterior algebra over its space of primitive elements will be borne out in the remaining sections where we see how the map ψ_r (resp. ψ^r) relates the primitive elements and their Pontrjagin products (resp. cup products) with the particular elements $B(h_\beta)$, $\beta \in \Omega_0$, of B(H).

4. Let $U' \subset U$ be a subspace of dimension m and let U'' be such that $U = U' + U''$ is a direct sum.

Now let $\mathcal{G}' = \mathcal{G}(U')$ and let

$$\varphi : \mathcal{G}' \to \mathcal{G}$$

be the isomorphism defined by letting $\varphi(A') \in \mathcal{G}$ equal A' on U' and vanish on U'' for all $A' \in \mathcal{G}'$. Hereafter we adopt the convention of using a prime on all previous notation when the reference is to U' instead of U.

Now regarding $^{r}U'$ as a subspace of ^{r}U it is clear that $^{r}U'$ is an invariant subspace of B(I). One sees immediately that the restriction of B(I) to $^{r}U'$ induces nothing more than the representation B' of I. We now have

Lemma 5.4. *Let* $f \in H$ *and let*

$$\psi^r(B(f)) = \underline{a}$$

and

$$\psi'^r(B'(f)) = \underline{a}';$$

then

$$\varphi^r(\underline{a}) = \underline{a}'.$$

Proof: If $\underline{u} \in H_r(\sigma')$ then clearly

$$\langle \underline{a}, \varphi_r(\underline{u}) \rangle = (B(f), \psi_r(\underline{u})).$$

But the range of $\psi_r(\underline{u})$ is contained in the subspace ${}^rU'$. Hence

$$(B(f), \psi_r(\underline{u})) = (B'(f), \psi_r(\underline{u})) = \langle \underline{a}', \underline{u} \rangle.$$

Thus $\langle \underline{a}, \varphi_r(\underline{u}) \rangle = \langle \underline{a}', \underline{u} \rangle$. Q.E.D.

Since, among other things, we are in the business of proving equivalences we will not yet assume knowledge of the Frobenius theorem. From §4.7 we know, without the Frobenius theorem, that the elements $h^o_\beta \in I_\beta$ defined by

$$h^o_\beta = \frac{1}{\sqrt{\mu s}} (\chi^{(1)}_\beta - \chi^{(-1)}_\beta) \qquad (5.4.1)$$

form a basis of H.

Now let ${}^nH$ be the kernel of B. It is immediately clear from Lemma 4.8 that $h^o_\beta \in {}^nH$ if and only if $\ell(\beta) > n$ and that, in fact, all such h^o_β form a basis of ${}^nH$.

We now have the following theorem on primitivity.

Theorem 5.4. *Assume* $r = 2p - 1$, $p > 1$. *Let* $\underline{b}_r \in H^r(\sigma)$

be defined by

$$\underline{b}_r = \psi^r(B(h^o_\epsilon))$$

where $\epsilon = \epsilon(r)$ is given by (4.8.2) and h^o_ϵ is defined as above. Then $\underline{b}_r \in P^r(\mathfrak{g})$ and $\underline{b}_r$ vanishes if and only if $p > n$.

Proof: Since $\ell(\epsilon) = p$ it follows from (5.4.1) $h^o_\epsilon \in {}^nH$ if and only if $p > n$. Hence $\underline{b}_r = 0$ if and only if $p > n$. Thus we may assume $p \geq n$.

Let $m = p - 1$ and let U' be chosen as in §5.4. Then $h^o_\epsilon \in {}^mH$ and hence from Lemma 5.4

$$\varphi^r(\underline{b}_r) = 0.$$

Now observe the uniqueness of $\epsilon = \epsilon(r)$ in the following regard. If $s \leq r$ and $\beta = \beta(s)$ is any self-associated partition of s, $\ell(\beta) \leq p$ where the equality sign holds if and only if $s = r$ and $\beta = \epsilon$. Thus if $\beta = \beta(s) \neq \epsilon(r)$ then $\ell(\beta) \leq m$ and hence $h^o_\epsilon \notin {}^mH_s$. Consequently by Lemma 5.4

$$\varphi^s(\underline{a}^s) \neq 0$$

where $\underline{a}^s = \psi(B(h^o_\beta))$. In other words, the kernel of $\varphi^s = 0$ for $s < r$ and the kernel of φ^r is $(\underline{b}^r)$. We now apply the criterion Lemma 2.6 and conclude that $\underline{b}^r \in P^r(\mathfrak{g})$. Q.E.D.

5. Having obtained Theorem 5.4 we now observe that the Frobenius theorem immediately yields the formula of Dynkin. Indeed, the Frobenius theorem asserts $h^o_\epsilon = h_\epsilon$. But then Theorem 5.4 asserts that the element $b_r \in J^r(\mathfrak{g})$ defined by

$$\langle b_r, u \rangle = (\eta^r(u), B(h_\epsilon)) \qquad (5.5.1)$$

vanishes if and only if $p > n$, where $r = 2p - 1$, and for $p < n$

is the invariant cocycle in the primitive class $\underline{b}_r$. But now if $u = A_1 \wedge \cdots \wedge A_r$ then by (4.9.3)

$$\langle b_r, A_1 \wedge A_2 \wedge \cdots \wedge A_r \rangle = c \cdot tr[A_1 A_2 \cdots A_r]$$

for a non-zero scalar c. This (up to scalar multiple) is the formula of Dynkin.

A second consequence of an application of the Frobenius theorem is the following sharpening of a result of Weyl. In [5], pp. 235-237, Weyl shows only that if $p > n$ the class $\underline{b}_r$ corresponding to the multilinear form b_r defined by (5.5.1) depends upon the classes $\underline{b}_s$ and their cup products, writing $s = 2q - 1$, $q \leqq n$. Here we have already seen that $\underline{b}_r = 0$ if $p > n$.

Until now we have been concerned with proving the equivalence of I, II, and III of §1.2. Here we will give a direct proof that $h^o_\epsilon = h_\epsilon$.

6. Let $p + q = r$. Define $\mathcal{S}(p) \subset \mathcal{S}$ by

$$\mathcal{S}(p) = \left\{ \sigma \in \mathcal{S} \;\middle|\; \begin{array}{l} \text{(a) } 1 \leqq i < j \leqq p \text{ implies } \sigma(i) < \sigma(j) \\ \text{(b) } p < i < j \leqq r \text{ implies } \sigma(i) < \sigma(j) \end{array} \right\}$$

Now if E is a vector space, E^* its dual, $c_p \in \bigwedge^p E^*$ and $c_q \in \bigwedge^q E^*$ then it is a familiar fact that

$$\langle c_p \wedge c_q, x_1 \wedge x_2 \wedge \cdots \wedge x_r \rangle$$

$$\sum_{\sigma \in \mathcal{S}(p)} sg(\sigma) \langle c_p, x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)} \rangle$$

$$\langle c_q, x_{\sigma(p+1)} \cdots x_{\sigma(r)} \rangle. \qquad (5.6.1)$$

Now let $\sigma \in \mathcal{S}_p$, $\tau \in \mathcal{S}_q$ and let $b_\sigma \in J^p(\mathcal{J})$ and $b_\tau \in J^q(\mathcal{J})$ be defined by

$$b_\sigma = \psi^p(B(h^\sigma)), \quad b_\tau = \psi^p(B(h^\tau)).$$

Also let $f \in H_r$ be defined by $\psi^r(B(f)) = b_\sigma \wedge b_\tau$. Then from (5.6.1)

$$(B(f), A_1 \otimes A_2 \otimes \cdots \otimes A_r)$$

$$= \sum_{\rho \in \mathcal{S}(p)} sg(\rho)(B(h^\sigma), A_{\rho(1)} \otimes \cdots \otimes A_{\rho(p)})$$

$$(B(h^\tau), A_{\rho(p+1)} \otimes \cdots \otimes A_{\rho(r)})$$

$$\sum_{\rho \in \mathcal{S}(p)} sg(\rho)(B(\sigma), \sum_{\nu \in \mathcal{S}_p} sg(\nu) A_{\rho\nu(1)} \otimes \cdots \otimes A_{\rho\nu(p)})$$

$$\cdot (B(\tau), \sum_{\mu \in \mathcal{S}_q} sg(\mu)\ A_{\rho(p+\mu(1))} \otimes \cdots \otimes A_{\rho(p+\mu(q))})$$

$$= (B(\hat{\sigma}\hat{\tau}), \sum_{\rho \in \mathcal{S}_r} sg(\rho) A_{\rho(1)} \otimes \cdots \otimes A_{\rho(r)})$$

where $\hat{\sigma}, \hat{\tau} \in \mathcal{S}_r$ are defined by $\hat{\sigma}(j) = \sigma(j)$, $1 \leq j \leq p$, and $\hat{\sigma}(j) = j$ for $j > p$; $\hat{\tau}(j) = p + \tau(j-p)$ for $j > p$ and $\hat{\tau}(j) = j$ for $1 \leq j \leq p$. Thus f can be chosen so that $f = h^{\hat{\sigma}\hat{\tau}}$.

Similarly let $u_\sigma \in J_p(\mathcal{J})$ and $u_\tau \in J_q(\mathcal{J})$ be such that

$$\psi_p(u_\sigma) = B(h^\sigma)$$

and

$$\psi_q(u_\tau) = B(h^\tau).$$

Then since

$$\eta^r(\sum_{\rho \in \mathcal{S}} sg(\rho)(A_{\rho(1)} \otimes \cdots \otimes A_{\rho(r)})) = a_1 \wedge \cdots \wedge a_r$$

where $a_i = \eta^1(A_i) \in C^1(\mathcal{J})$, $i = 1, 2, \cdots, r$, (see (5.2.2)) by the same argument as above

$$\psi_r(u_\sigma \wedge u_\tau) = \frac{p!q!}{r!} B(h^{\hat{\sigma}\hat{\tau}}).$$

Now recalling the role played by Δ as given in §4.6 we obtain the following information on cup and Pontrjagin products.

Lemma 5.6. Let $p + q = r$. Let $\beta_1 \in \Omega_o(p)$ and $\beta_2 \in \Omega_o(q)$. Let $u_p \in H_p(\mathcal{J})$, $u_q \in H_q(\mathcal{J})$ be such that

$$\psi_p((u_p)) = B((h_{\beta_1})), \quad \psi_q((u_q)) = B((h_{\beta_2}))$$

and let $a_p \in H^p(\mathcal{J})$ and $a_q \in H^q(\mathcal{J})$ be such that

$$\psi^p(B((h_{\beta_1}))) = (a_p), \quad \psi^q(B((h_{\beta_2}))) = (a_q).$$

Let $\delta = \delta(\beta_1) + \delta(\beta_2) \in \Omega(r)$ be the partition sums. Then by §4.6 if $\delta \notin \Delta(r)$, $u_p \cup u_q = 0$ and $a_p \cup a_q = 0$. On the other hand, if $\delta \in \Delta(r)$ and $\beta \in \Omega_o$ is such that $\mathcal{S}(\beta) = \delta$ then

$$\psi_r((u_p \cup u_q)) = B((h_\beta))$$

and

$$\psi^r(B((h_\beta))) = (a_p \cup a_q)$$

Now let $r = 2p - 1$ where $p \leq n$. Let $\beta \in \Omega_o$ and let $u \in H_r(\mathcal{J})$ be such that $\psi_r(u) = B(h_\beta)$. Then if $\beta \neq \epsilon$ it follows immediately from Lemma 5.6 that $u \in D_r(\mathcal{J})$. On the other hand, we have already proved (Theorem 5.4)

$$\psi^r(B(h^o_\epsilon)) \in P^r(\mathcal{J}).$$

Thus we have the following orthogonality relation:

$$(B(h_\epsilon^o),\ B(h_\beta)) = 0 \qquad (5.6.1)$$

for all $\beta \in \Omega_o$, where $\beta \neq \epsilon$.

Now observe from (5.4.1)

$$(h_{\beta_1}^o \cdot h_{\beta_2}^o) = \begin{cases} 0 & \text{if } \beta_1 \neq \beta_2 \\ (e_{\beta_1}) & \text{if } \beta_1 = \beta_2 \end{cases}$$

Hence upon writing (see §4.7) $h_\beta = \sum_{\beta'} a_{\beta'} h_{\beta'}^o$, it follows that

$$(h_\beta \cdot h_\epsilon^o) = a_\epsilon (e_\epsilon).$$

Since

$$\text{tr } B(h_\beta \cdot h_\epsilon^o) = (B(h_\beta),\ B(h_\epsilon^o))$$

it follows then from (5.6.1) that $a_\epsilon = 0$ since tr $B(e_\epsilon) \neq 0$. Thus if $\{f,g\}$ designates the usual inner product (in I),

$$\{f,g\} = \sum_{\sigma \in \delta} f(\sigma)\overline{g(\sigma)}$$

for $f, g \in I$, it follows that $\{h_\beta, h_\epsilon^o\} = 0$ if $\beta \neq \epsilon$. But now expanding h_ϵ^o in terms of the basis $\{h_\beta\}$,

$$h_\epsilon^o = \sum c_\beta h_\beta,$$

and observing that the h_β are obviously mutually orthogonal in I it follows that $c_\beta = 0$ for all $\beta \neq \epsilon$. Hence

$$h_\epsilon^o = c_\epsilon h_\epsilon$$

so that $h_\epsilon \quad I_\epsilon$. Thus we have proved $h_\epsilon \quad I_\epsilon$ directly.

If $r = 2p - 1$ and $\underline{u}_r \in H_r(\mathcal{J})$ is such that $\eta_r(\underline{u}_r) = B(h_\epsilon)$ note that Lemma 5.6 and the Frobenius theorem imply that .

$\underline{u}_r \in P_r(\mathcal{J})$. Finally, turning our attention to the general element $\beta \in \Omega_o$, Lemma 5.6 yields

Theorem 5.6. *Let $\beta \in \Omega_o(r)$ and let $\delta_1, \delta_2, \cdots, \delta_p$ be the lengths of the "shells" of the graph $G(\beta)$ of β (see §4.5). If $\underline{b}_\beta \in H^r(\mathcal{J})$ is defined by*

$$\underline{b}_\beta = \psi^r(B(h_\beta))$$

then $\underline{b}_\beta$ vanishes if and only if $\delta_1 > 2n - 1$. Moreover,

$$(\underline{b}_\beta) = (\underline{b}_{\delta_1} \cup \underline{b}_{\delta_2} \cup \cdots \cup \underline{b}_{\delta_p})$$

where $\underline{b}_{\delta_i}$ is a primitive δ_i-cohomology class.

Clearly, the analogous homological statement is also true.

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